

# On the consistency of Constraints in Field Theories

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## Abstract

We consider how the principles of causality and equivalence restrict the background in which matter field theories are defined; because these matter field theories have to respect those constraints, then they will result to be inconsistent in general.

## Introduction

The Principles of Causality and Equivalence are the two fundamental principles that define the properties of the geometry: the former requires the space to be a spacetime with definite structural properties, leading to relativity, and the latter requires this spacetime to be a curved spacetime with definite dynamical properties, leading to relativistic gravitation; the consequence of having a background with such restrictions is that matter fields propagating in this background will necessarily behave in a correspondingly restricted manner: that is among all the possible matter field theories that can be defined, only some will have propagation whose speed is always smaller than  $c$  and among these, only some will be coupled to a gravitational field which always permits the existence of a unique system of reference in local free fall. More in detail, the principle of causality requires the spacetime be a  $(1+3)$ -dimensional differential manifold in which a light-cone structure is defined by means of light-like vectors, which are defined only if lengths are maintained, and thus metric informations must be preserved by requiring what is known as metricity condition, as discussed by Hayashi in [1], or in other terms by Hehl, Von Der Heyde, Kerlick and Nester in [2]; on the other hand, the principle of equivalence requires this  $(1+3)$ -dimensional manifold to have connections whose symmetric part, the one containing gravitational informations, has to be uniquely defined by demanding again the metricity condition but with in addition the complete antisymmetry of the Cartan torsion tensor of that connection, as it has been explained elsewhere in [3].

After such a restriction for the connection, the same restriction will feature the covariant derivatives and hence the propagation of the matter field; indeed as soon as non trivial connections are considered, for instance in presence of electrodynamics, conditions may occur for which causality is violated, in what is known to be the Velo-Zwanziger problem, discussed first by Velo and Zwanziger in [4] and [5], further generalized to include gravitation, as discussed by Deser

and Waldron in [6]; here the extension to the torsional case will be considered. For those matter fields whose propagation is free of problems, and whose matter field equations are perfectly defined, we will further consider the whole system of field equations, for which torsion couples to spin, so that the complete antisymmetry of torsion gives the complete antisymmetry of spin, resulting in constraints, as already discussed elsewhere in [7]; here these constraints will be taken into account for the counting of degrees of freedom needed to define the matter field itself.

A discussion about the consequences of these restriction will be carried out with several examples.

## 1 Foundations of the Geometry

In a given geometry, the metric structure is given in terms of two symmetric metric tensors  $g_{\alpha\beta}$  and  $g^{\alpha\beta}$  that are one the inverse of the other, and differential operations  $D_\mu$  are defined through the connections  $\Gamma_{\alpha\beta}^\rho$ ; the principle of causality is implemented by requiring that the metric can locally be reduced to the Minkowskian form of signature  $(1, -1, -1, -1)$ , and that the covariant derivatives applied upon the metric tensors vanish according to what is called metricity condition  $D_\mu g = 0$ , as discussed in [1], or as explained in other terms in [2]. The principle of equivalence is implemented by demanding again the condition of metricity but now along with the complete antisymmetry of Cartan torsion tensor  $Q_{\alpha\mu\rho}$ , as explained in [3].

In this background, we will define Riemann curvature tensor  $G_{\alpha\beta\mu\nu}$  as

$$G_{\lambda\mu\nu}^\alpha = \partial_\mu \Gamma_{\lambda\nu}^\alpha - \partial_\nu \Gamma_{\lambda\mu}^\alpha + \Gamma_{\rho\mu}^\alpha \Gamma_{\lambda\nu}^\rho - \Gamma_{\rho\nu}^\alpha \Gamma_{\lambda\mu}^\rho \quad (1)$$

antisymmetric in both the first and the second couple of indices, allowing only one independent contraction, Ricci curvature tensor  $G_{\alpha\lambda\beta}^\lambda = G_{\alpha\beta}$ , whose contraction is Ricci curvature scalar  $G_{\alpha\beta} g^{\alpha\beta} = G$  and this will set our convention.

Riemann curvature tensor, Ricci curvature tensor and scalar, together with Cartan torsion tensor verify

$$D_\rho Q^{\rho\mu\nu} + \left( G^{\nu\mu} - \frac{1}{2} g^{\nu\mu} G \right) - \left( G^{\mu\nu} - \frac{1}{2} g^{\mu\nu} G \right) \equiv 0 \quad (2)$$

and

$$D_\mu \left( G^{\mu\rho} - \frac{1}{2} g^{\mu\rho} G \right) - \left( G_{\mu\beta} - \frac{1}{2} g_{\mu\beta} G \right) Q^{\beta\mu\rho} + \frac{1}{2} G^{\mu\kappa\beta\rho} Q_{\beta\mu\kappa} \equiv 0 \quad (3)$$

which are geometric identities in the form of conservation laws, called Jacobi-Bianchi identities.

We remark that from the metric tensor it is possible to define the Levi-Civita tensor  $\varepsilon$  for which  $D_\mu \varepsilon = 0$  precisely because of the complete antisymmetry of torsion.

In turn, since torsion is completely antisymmetric then we can write

$$Q^{\beta\mu\rho} = \varepsilon^{\beta\mu\rho\sigma} W_\sigma \quad (4)$$

in terms of what is called axial torsion vector.

Next we consider the fact that in the case of complex fields also gauge covariance needs to be considered, and in this case differential operations  $D_\mu$  are defined through the gauge connections  $A_\alpha$  so to give gauge covariant derivatives, as known.

In this background, we define Maxwell curvature tensor  $F_{\mu\nu}$  as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (5)$$

antisymmetric, and thus irreducible.

The Maxwell curvature tensor verifies

$$\partial_\alpha F_{\mu\sigma} + \partial_\sigma F_{\alpha\mu} + \partial_\mu F_{\sigma\alpha} \equiv 0 \quad (6)$$

which are geometric identities known as Jacobi-Cauchy identities, and also the commutator of covariant derivatives applied upon Maxwell curvature tensor gives

$$D_\rho \left( D_\sigma F^{\sigma\rho} + \frac{1}{2} F_{\alpha\mu} Q^{\alpha\mu\rho} \right) \equiv 0 \quad (7)$$

in the form of conservation laws.

We shall now address the fundamental issue based on the fact that, although in the case of the spacetime curvature (1) the object upon which the ordinary derivatives act is a connection and thus it can not be generalized in order to be written in terms of covariant derivatives, nevertheless in the case of the gauge curvature (5) the object upon which the ordinary derivatives act is a vector and so it could be generalized in order to be written in terms of covariant derivatives as

$$\Phi_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + A_\rho Q^\rho_{\mu\nu} = F_{\mu\nu} + A_\rho Q^\rho_{\mu\nu} \quad (8)$$

hence creating the problem of which Maxwell curvature should be considered: to solve this ambiguity it is enough to see that even if (5) *could* be generalized up to (8), such a generalization is not needed since (5) is *already* the most general definition that can be given in order for Maxwell curvature tensor to be the commutator of gauge covariant derivatives, exactly in the same way in which Riemann curvature tensor is the commutator of covariant derivatives.

Within this background, to define matter fields that can be classified according to the value of their spin we have to consider that a given matter field of spin  $s$  possesses  $2s + 1$  degrees of freedom, which have to correspond to the  $2s + 1$  independent solutions of a system of equations that specify the highest-order time derivative for all components of the field, called system of matter field equations.

However, since it may happen that field equations are not enough to determine the correct rank of the solution, restrictions need be imposed in terms of equations in which all components of the field have highest-order time derivatives that never occur, called constraints; these constraints can be imposed in two ways, either being implied by the field equations, or being assigned as subsidiary conditions that come along with the field equations themselves.

Although the former procedure seems more elegant, whenever interactions are present it can give rise to two types of problems, the first of which concerning the fact that the presence of the interacting fields could increase the

order derivative of the constraining equation up to the same order derivative of the field equations themselves, creating the possibility that highest-order time derivatives of some component occur, converting the constraint into a field equation, then spoiling the counting of degrees of freedom.

Before proceeding we have to remind the reader that to check causal propagation, the general method is to consider in the field equations eventually modified by constraints the terms of the highest-order derivative of the field, formally replacing the derivatives with the vector  $n$  in order to obtain the propagator, of which one has to compute the determinant setting it to zero in order to get an equation in terms of  $n$  called characteristic equation, whose solutions are the normal to the characteristic surfaces, representing the propagation of the wave fronts: if there is no time-like normal among all the possible solutions, then there is no space-like characteristic surface, and therefore there is no acausal propagation of the wave front.

If in the constraining equation the highest-order time derivative never appeared, or if it actually appeared but could be removed by means of field equations, then the constraint is a constraint indeed, but in this case a second type of problem can arise, regarding the fact that the interacting fields could let appear terms of the highest-order derivative in the propagator, allowing these terms to influence the propagation of the wave fronts themselves.

Once this analysis is performed, causal propagation of wave fronts is checked, and the exact number of degrees of freedom of the matter field solution is established, the last requirement for this system of matter field equations is that they have to ensure the complete antisymmetry of the spin, so that taking the spin  $S^{\nu\sigma\rho}$  with the energy  $T^{\sigma\rho}$  and also the current  $J^\mu$  they have to be such that the relationships

$$D_\rho S^{\rho\mu\nu} + \frac{1}{2} (T^{\mu\nu} - T^{\nu\mu}) = 0 \quad (9)$$

and

$$D_\mu T^{\mu\rho} - T_{\mu\beta} Q^{\beta\mu\rho} - S_{\beta\mu\kappa} G^{\mu\kappa\beta\rho} + J_\beta F^{\beta\rho} = 0 \quad (10)$$

and also

$$D_\rho J^\rho = 0 \quad (11)$$

are verified, implying the whole set of field equations

$$D_\sigma F^{\sigma\rho} + \frac{1}{2} F_{\alpha\mu} Q^{\alpha\mu\rho} = J^\rho \quad (12)$$

and also

$$\left( G^{\sigma\rho} - \frac{1}{2} g^{\sigma\rho} G \right) + \frac{1}{2} \left( \frac{1}{4} g^{\sigma\rho} F^2 - F^{\sigma\mu} F^\rho{}_\mu \right) = -\frac{1}{2} T^{\sigma\rho} \quad (13)$$

and

$$Q^{\nu\sigma\rho} = S^{\nu\sigma\rho} \quad (14)$$

to be such that the conservation laws (2) and (3) and also (7) are satisfied automatically.

This determines the set-up of the fundamental field equations in minimal coupling, that is taking the least-order derivative possible in both sides of the field equations.

Clearly, also the variational method could be employed with actions in minimal coupling, that is least-order derivative terms in the action; obviously the two methods leads to the very same result.

## 2 Geometrically Constrained Matter Fields

Having settled the background in this way, we begin to consider the issue of which matter fields could actually be defined in it.

### 2.1 Light-Like Characteristic Surfaces and Completely Antisymmetric Torsion

Clearly, because the background is characterized by these restrictions, then matter fields will behave in a correspondingly restricted way.

We will now consider examples of matter fields discussing how they can be defined as to respect these restrictions:

**Vector Fields.** In the case of a vector  $V_\mu$  it is possible to define, beside the standard covariant derivative given in terms of the connection, another more special differential operation given in terms of no additional field whose form is given by  $Z_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$  called exterior derivative; although this form of the derivative would be compulsory for gauge fields, for a massive vector field there is no gauge symmetry that obliges the derivative to be the exterior derivative, and although the exterior derivative would still be interesting due to its special form, there is of course the possibility to use the most general covariant derivative as defined in the standard way.

In the following we will consider a massive vector field whose dynamical term is written in terms of covariant derivative, first in the special case of a derivative which is formally the exterior derivative but with respect to the most general connection given by the expression  $Z_{\rho\mu} = D_\rho V_\mu - D_\mu V_\rho$ , and later by the most general covariant derivative  $D_\rho V_\mu$  as in the standard form.

So given the couple of complex conjugate vector fields  $V_\mu$  and  $V_\mu^*$ , we postulate the most general Proca Lagrangian

$$L = G - \frac{1}{4}F_{\alpha\mu}F^{\alpha\mu} - \frac{1}{2}Z_{\alpha\mu}^*Z^{\alpha\mu} - \frac{\lambda}{4}Z_{\alpha\beta}^*Z_{\mu\nu}\varepsilon^{\alpha\beta\mu\nu} + m^2V_\mu^*V^\mu \quad (15)$$

in terms of the parameters  $\lambda$  and  $m$ , which will have to be varied to yield the field equations.

Then we have that Proca matter field equations are

$$D_\mu Z^{\mu\alpha} + \frac{\lambda}{2}D_\mu Z_{\eta\rho}\varepsilon^{\mu\eta\rho\alpha} + m^2V^\alpha = 0 \quad (16)$$

which specify the second-order time derivative for only the spatial components, but which also develop the constraint

$$\begin{aligned} & m^2D_\mu V^\mu - \frac{\lambda}{4}Q_{\rho\mu\nu}D^\rho Z_{\alpha\beta}\varepsilon^{\alpha\beta\mu\nu} - \frac{1}{2}Q^{\rho\alpha\beta}D_\rho Z_{\alpha\beta} - \\ & - \frac{\lambda}{2}G^\rho{}_{\beta\mu\nu}Z_{\rho\alpha}\varepsilon^{\alpha\beta\mu\nu} - G^{\alpha\beta}Z_{\alpha\beta} - \frac{i}{2}F^{\alpha\beta}Z_{\alpha\beta} - \frac{i\lambda}{4}F_{\alpha\beta}Z_{\mu\nu}\varepsilon^{\alpha\beta\mu\nu} = 0 \end{aligned} \quad (17)$$

and we see that due to the presence of torsion this constraint contains terms with the second-order time derivative of spatial components, which can anyway be removed by means of field equations, and thus it is a constraint indeed; this constraint can be used to allow field equations to specify the second-order time derivative of all components, although by plugging this constraint back into the field equations again the presence of torsion would give rise to a third-order derivative field equation, and thus a different procedure must be followed.

We can proceed by separating the variables of the vector field according to the decomposition given by  $V_\mu = U_\mu + D_\mu B$  with  $D_\mu U^\mu = 0$ , so that after this decomposition the system of the constraint with the system of field equations is correspondingly decomposed as

$$\begin{aligned}
& -\frac{1}{2}Q_{\rho\alpha\beta}Q^{\rho\alpha\sigma}D_\sigma D^\beta B - \frac{\lambda}{4}Q_{\sigma\alpha\beta}Q_{\rho\mu\nu}D^\rho D^\sigma B\varepsilon^{\alpha\beta\mu\nu} + m^2 D^2 B - \\
& -\frac{\lambda}{2}Q_{\rho\mu\nu}D^\rho D_\alpha U_\beta \varepsilon^{\alpha\beta\mu\nu} - \frac{1}{2}D_\rho Q^{\rho\alpha\beta}Q_{\sigma\alpha\beta}D^\sigma B + \frac{1}{2}Q^{\rho\alpha\beta}G_{\sigma\rho\alpha\beta}D^\sigma B - \\
& -iQ^{\rho\alpha\beta}F_{\rho\alpha}D_\beta B - \frac{\lambda}{4}Q_{\rho\mu\nu}D^\rho Q_{\sigma\alpha\beta}D^\sigma B\varepsilon^{\alpha\beta\mu\nu} + \frac{\lambda}{2}Q_{\sigma\alpha\rho}G^\rho_{\beta\mu\nu}D^\sigma B\varepsilon^{\alpha\beta\mu\nu} - \\
& -\frac{i\lambda}{2}Q_{\rho\mu\nu}F_{\alpha\beta}D^\rho B\varepsilon^{\alpha\beta\mu\nu} - \frac{1}{2}Q_{\rho\alpha\beta}Q^{\rho\alpha\sigma}D_\sigma U^\beta - D_\rho Q^{\rho\alpha\beta}D_\alpha U_\beta - \\
& -iF^{\alpha\beta}D_\alpha U_\beta + \frac{\lambda}{2}G^\rho_{\beta\mu\nu}D_\alpha U_\rho \varepsilon^{\alpha\beta\mu\nu} - \frac{\lambda}{2}G^\rho_{\beta\mu\nu}D_\rho U_\alpha \varepsilon^{\alpha\beta\mu\nu} - \\
& -\frac{i\lambda}{2}F_{\alpha\beta}D_\mu U_\nu \varepsilon^{\alpha\beta\mu\nu} - \frac{i}{2}D_\rho Q^{\rho\alpha\beta}F_{\alpha\beta}B - \frac{i}{2}Q^{\rho\alpha\beta}D_\rho F_{\alpha\beta}B + \\
& +\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta}B - \frac{i\lambda}{4}Q_{\rho\mu\nu}D^\rho F_{\alpha\beta}B\varepsilon^{\alpha\beta\mu\nu} + \frac{i\lambda}{2}G^\rho_{\beta\mu\nu}F_{\alpha\rho}B\varepsilon^{\alpha\beta\mu\nu} + \\
& +\frac{\lambda}{4}F_{\alpha\beta}F_{\mu\nu}B\varepsilon^{\alpha\beta\mu\nu} + \frac{1}{2}Q^{\rho\alpha\beta}G_{\sigma\rho\alpha\beta}U^\sigma - \frac{i}{2}Q^{\rho\alpha\beta}F_{\rho\alpha}U_\beta = 0 \tag{18}
\end{aligned}$$

$$\begin{aligned}
& \frac{\lambda}{2}Q_{\sigma\mu\eta}D_\rho D^\sigma B\varepsilon^{\mu\eta\rho\alpha} + D^2 U^\alpha - \frac{1}{2}Q^{\mu\sigma\alpha}Q_{\mu\sigma\rho}D^\rho B - D_\mu Q^{\mu\sigma\alpha}D_\sigma B + \\
& +iF^{\mu\alpha}D_\mu B - \frac{\lambda}{2}G_{\sigma\mu\eta\rho}D^\sigma B\varepsilon^{\mu\eta\rho\alpha} + \frac{i\lambda}{2}F_{\mu\eta}D_\rho B\varepsilon^{\mu\eta\rho\alpha} + m^2 D^\alpha B - \\
& -Q^{\sigma\mu\alpha}D_\sigma U_\mu + \frac{\lambda}{2}Q_{\sigma\mu\eta}D^\sigma U_\rho \varepsilon^{\mu\eta\rho\alpha} - \frac{i}{2}Q^{\mu\sigma\alpha}F_{\mu\sigma}B + iD_\mu F^{\mu\alpha}B + \\
& +\frac{i\lambda}{2}D_\rho F_{\mu\eta}B\varepsilon^{\mu\eta\rho\alpha} - G^{\mu\alpha}U_\mu + iF^{\alpha\mu}U_\mu - \frac{\lambda}{2}G_{\sigma\mu\eta\rho}U^\sigma \varepsilon^{\mu\eta\rho\alpha} + \\
& +\frac{i\lambda}{2}F_{\mu\eta}U_\rho \varepsilon^{\mu\eta\rho\alpha} + m^2 U^\alpha = 0 \tag{19}
\end{aligned}$$

which specifies the second-order time derivative of all components.

In this equation we take the derivatives and we separate the contribution of torsion, the gauge field and the metric so to write it with respect to the partial derivatives which will be formally replaced with the vector  $n$  giving the propagator

$$P(n) = \left[ \frac{n^2 (m^2 - W^2) + (n \cdot W)^2}{\lambda (W^\alpha n^2 - n \cdot W n^\alpha)} \middle| \frac{-\lambda (W_\beta n^2 - n \cdot W n_\beta)}{n^2 g^\alpha_\beta} \right] \tag{20}$$

whose determinant equal to zero gives the characteristic equation

$$n^2 m^2 - (1 - \lambda^2) n^2 W^2 + (1 - \lambda^2) (n \cdot W)^2 = 0 \tag{21}$$

that will have to be discussed.

To begin with, we see that we can avoid time-like solutions of the characteristic equation only if we have that  $\frac{m^2}{\lambda^2 - 1} \leq -W^2$  or in the case  $|\lambda| < 1$  involving the parameter  $\lambda$  only; since the first strong torsion condition fails to hold in the always possible weak torsion situation, then  $0 \leq |\lambda| \leq 1$  is the only case left.

In this case, it is worth discussing the  $|\lambda| = 1$  model on its own: we see that the characteristic equation reduces to have solutions of the light-like type alone with no further constraints for torsion to be imposed.

We finally turn to the discussion of the  $0 \leq |\lambda| < 1$  case: we see that the condition  $0 \leq m^2 - (1 - \lambda^2) W^2$  coming from the requirement of reality of the solutions of this characteristic equation implies  $n^2 \leq 0$  so that space-like as well as light-like solutions alone are possible; however this condition is a constraint imposed upon the value torsion can possibly have.

The special case given by  $|\lambda| = 0$  has no particular issue, but it deserves to be studied independently in order to better underline the differences in the methods followed to investigate the field equations.

So given the couple of complex conjugate vector fields  $V_\mu$  and  $V_\mu^*$ , we postulate the special Proca Lagrangian

$$L = G - \frac{1}{4} F_{\alpha\mu} F^{\alpha\mu} - \frac{1}{2} Z_{\alpha\mu}^* Z^{\alpha\mu} + m^2 V_\mu^* V^\mu \quad (22)$$

in terms of the parameter  $m$ , which will have to be varied to yield the field equations.

Then we have that Proca matter field equations are

$$D_\mu Z^{\mu\alpha} + m^2 V^\alpha = 0 \quad (23)$$

which specify the second-order time derivative for only the spatial components, but which also develop the constraint

$$2m^2 D_\mu V^\mu - i F^{\alpha\beta} Z_{\alpha\beta} - 2G^{\alpha\beta} Z_{\alpha\beta} - Q^{\rho\alpha\beta} D_\rho Z_{\alpha\beta} = 0 \quad (24)$$

and we see that this constraint does not contain terms with the second-order time derivative of spatial components, and thus it is a constraint; this constraint can be plugged back into the original field equations and due to the presence of torsion a third-order derivative term would appear in the field equations, although the fact that torsion is completely antisymmetric allows the removal of this third-order derivative term leaving the field equations

$$\begin{aligned} & -\frac{1}{m^2} D^\rho Q_{\rho\alpha\beta} D^\nu D^\alpha V^\beta - \frac{1}{2m^2} Q^{\rho\alpha\beta} Q_{\rho\alpha\sigma} D^\nu D^\sigma V_\beta - \frac{i}{m^2} F_{\alpha\beta} D^\nu D^\alpha V^\beta + \\ & + D^2 V^\nu - \frac{1}{m^2} D^\nu D_\rho Q^{\rho\alpha\beta} D_\alpha V_\beta - \frac{1}{2m^2} Q_{\rho\alpha\beta} D^\nu Q^{\rho\alpha\sigma} D_\sigma V^\beta - \\ & - \frac{1}{2m^2} D^\nu Q^{\rho\alpha\beta} Q_{\rho\alpha\sigma} D^\sigma V_\beta + \frac{1}{2m^2} D^\nu Q^{\beta\rho\alpha} G_{\sigma\beta\rho\alpha} V^\sigma - \frac{i}{2m^2} D^\nu Q^{\rho\alpha\beta} F_{\rho\alpha} V_\beta + \\ & + \frac{1}{2m^2} Q^{\alpha\beta\rho} G_{\sigma\alpha\beta\rho} D^\nu V^\sigma - \frac{i}{2m^2} Q_{\rho\alpha\beta} F^{\rho\alpha} D^\nu V^\beta - Q^{\alpha\beta\nu} D_\alpha V_\beta - \\ & - \frac{i}{m^2} D^\nu F^{\alpha\beta} D_\alpha V_\beta + \frac{1}{2m^2} Q_{\alpha\beta\rho} D^\nu G^{\sigma\alpha\beta\rho} V_\sigma - \frac{i}{2m^2} Q_{\rho\alpha\beta} D^\nu F^{\rho\alpha} V^\beta - \\ & - G^{\mu\nu} V_\mu + i F^{\nu\beta} V_\beta + m^2 V^\nu = 0 \end{aligned} \quad (25)$$

which specify the second-order time derivative of all components.

Writing these field equations in the form in which the torsional, gauge and metric contributions have been separated apart we get the propagator

$$P(n) = [n^2 m^2 g^{\beta\nu} - n_\alpha (g^{\alpha\beta} W^2 - W^\alpha W^\beta + G^{\alpha\beta} - G^{\beta\alpha} + i F^{\alpha\beta}) n^\nu] \quad (26)$$

whose determinant equal to zero gives the characteristic equation

$$n^2 m^2 - n^2 W^2 + (n \cdot W)^2 = 0 \quad (27)$$

that will have to be discussed.

Following the same reasoning in this instance we see that the condition given by the relationship  $0 \leq m^2 - W^2$  implies that  $n^2 \leq 0$  and space-like or light-like

solutions alone are allowed; and again this condition is a constraint imposed upon the value torsion can assume.

This result is no different from that we obtained as the  $|\lambda| = 0$  case of the previous model; however, the methodology used to study these two cases was different: in the former case, the constraint was of the same order derivative of the field equations, giving higher order derivative terms than the leading ones in the field equations, whereas in this latter case the constraint was of lower order derivative than the field equations, and no higher order derivative term was found beside the leading ones in the field equations themselves. However, in both cases, highest-order derivative terms accompany the leading one in the field equations, so that terms depending on the Maxwell and Riemann curvature tensors as well as Cartan torsion tensor are now present in the propagator, although only torsion changes the structure of the characteristic equation, and although it is always possible to impose constraints on it in order to avoid to spoil the causal structure of the characteristic surfaces that are solutions of the characteristic equation itself.

We will now leave the treatment of the exterior derivatives, to turn our attention to the most general covariant derivative as defined in the standard way.

So given the vector field  $V_\mu$ , we postulate Fermi Lagrangian

$$L = G - \frac{1}{4}F_{\alpha\mu}F^{\alpha\mu} - D_\rho V_\mu^* D^\rho V^\mu + m^2 V_\mu^* V^\mu \quad (28)$$

in terms of the parameter  $m$ , which will have to be varied to yield the field equations.

Thus we have that Fermi matter field equations are

$$D^2 V^\alpha + m^2 V^\alpha = 0 \quad (29)$$

which specify the second-order time derivative of all components.

We clearly see that because Fermi field equations do not develop constraint, then the propagator has determinant equal to zero that gives the characteristic equation with solutions of the light-like type alone.

The difference between Proca and Fermi fields is that, whereas in the Proca cases measures had to be taken to keep the causal structure, for the Fermi field the causal structure of the characteristic surfaces that are solutions of the characteristic equations is always achieved. So to define the torsional vector field with no problems, neither in the propagation of the solutions, nor in the structure of the field equations, care must be taken in fine-tuning some parameter or the value of torsion, or to define the form of the field equations.

This concludes the exam of some cases of vector fields, for which the complex value of the field was simply imposed *ad hoc*, whereas their transformation law was still real; we will now turn our attention to a more serious description of matter fields: this description is entailed by spinor fields, that is complex fields that transform according to complex representations of the Lorentz group.

**Spinor Fields.** Spinor fields transform according to spinorial representation of the Lorentz group; a given spinorial transformation  $S$  can be expanded in infinitesimal generators of the form  $\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$  written in terms of the set



of 4-dimensional matrices verifying Clifford algebra  $\gamma^\mu$  from which the matrix  $\gamma = i\gamma^0\gamma^1\gamma^2\gamma^3$  is also defined.

The  $\gamma^0$  matrix is used to define complex conjugation, and differential operations are introduced through the spinorial connection as usual, and they are called spinorial derivatives.

**Vector-Spinor Fields.** The column of vectors  $\psi^\mu$  that transforms as  $\psi'^\mu = S^{\frac{\partial x'^\mu}{\partial x^\nu}}\psi^\nu$  defines the Rarita-Schwinger spinors.

Given the Rarita-Schwinger spinor, we postulate the Rarita-Schwinger Lagrangian

$$L = G - \frac{1}{4}F^2 - \frac{1}{2}\varepsilon^{\alpha\nu\rho\sigma}(\bar{\psi}_\alpha\gamma\gamma_\nu D_\rho\psi_\sigma - D_\rho\bar{\psi}_\alpha\gamma\gamma_\nu\psi_\sigma) - m\bar{\psi}_\mu\sigma^{\mu\nu}\psi_\nu \quad (30)$$

in terms of the parameter  $m$ , which will have to be varied to yield the field equations.

Then we have that Rarita-Schwinger spinor matter field equations are

$$\varepsilon^{\alpha\nu\rho\sigma}\gamma\gamma_\nu D_\rho\psi_\sigma + m\sigma^{\alpha\rho}\psi_\rho = 0 \quad (31)$$

which specify the time derivative for only the spatial components, but which also develop the constraint

$$3m^2\gamma_\nu\psi^\nu - 4i\varepsilon^{\nu\eta\rho\sigma}Q_{\kappa\eta\rho}\gamma\gamma_\nu D^\kappa\psi_\sigma - 2i\varepsilon^{\nu\eta\rho\sigma}G_{\alpha\kappa\eta\rho}\gamma\gamma_\nu\sigma^{\alpha\kappa}\psi_\sigma + 4i\varepsilon^{\nu\eta\rho\sigma}G_{\alpha\sigma\eta\rho}\gamma\gamma_\nu\psi^\alpha + 4\varepsilon^{\nu\eta\rho\sigma}F_{\eta\rho}\gamma\gamma_\nu\psi_\sigma = 0 \quad (32)$$

which due to the presence of torsion does contain terms with the time derivative of spatial components that can not be removed by using the field equations, and thus this constraint is lost because it is converted into a field equation.

Because this special form of the Rarita-Schwinger equations was given in order to imply those constraints that are not gotten any longer anyway then there is no need to consider it anymore, and so we will simply consider the Rarita-Schwinger equations as given in the original paper by Rarita and Schwinger.

Given the Rarita-Schwinger spinor, we postulate the Rarita-Schwinger Lagrangian

$$L = G - \frac{1}{4}F^2 + \frac{i}{2}(\bar{\psi}^\alpha\gamma^\rho D_\rho\psi_\alpha - D_\rho\bar{\psi}_\alpha\gamma^\rho\psi^\alpha) - m\bar{\psi}_\mu\sigma^{\mu\nu}\psi_\nu \quad (33)$$

in terms of the parameter  $m$ , which will have to be varied to yield the field equations.

Then we have that Rarita-Schwinger spinor matter field equations are

$$i\gamma^\mu D_\mu\psi^\rho - m\sigma^{\rho\alpha}\psi_\alpha = 0 \quad (34)$$

which specify the time derivative of all components.

In their original form the Rarita-Schwinger equations do not develop any constraint, their propagator has determinant equal to zero that gives the characteristic equation with solutions of the light-like type alone.

So in this case the causal structure of the characteristic surfaces that are solutions of the characteristic equations is always accomplished.

And so to define the torsional vector-spinor field with no problems for what concerns the number of degrees of freedom of the solutions, and hence for the

constraints that are defined as subsidiary conditions that come along with field equations, care must be taken in order to define both the constraints and the form of the field equations themselves.

We will finally turn our attention to the case of a special spinor field that is defined with no constraints needed.

**Scalar-Spinor Fields.** The column of scalars  $\psi$  transforming as  $\psi' = S\psi$  defines the Dirac spinors.

Given the Dirac spinor, we postulate the Dirac Lagrangian

$$L = G - \frac{1}{4}F^2 + \frac{i}{2}(\bar{\psi}\gamma^\rho D_\rho\psi - D_\rho\bar{\psi}\gamma^\rho\psi) - m\bar{\psi}\psi \quad (35)$$

in terms of the parameter  $m$ , which will have to be varied to yield the field equations.

Then we have that Dirac spinor matter field equations are

$$i\gamma^\mu D_\mu\psi - m\psi = 0 \quad (36)$$

which specify the time derivative of all components.

As it is well-known being Dirac equations (36) unconstrained, their propagator has determinant equal to zero that gives the characteristic equation with solutions of the light-like type alone.

Thus one can always properly define torsional scalar-spinor field with no problems in the structure of the field equations.

## 2.2 Completely Antisymmetric Spin

Thus far we have discussed in which ways the full connection with potentials of gauge, metric and torsion can affect the propagation of matter fields, and we have seen which effects are particularly due to the complete antisymmetry of torsion.

We are now going to turn our attention to the issue of the counting of degrees of freedom in terms of the constraints imposed by the completely antisymmetric spin.

**Vector Fields.** As before we will consider first the vector field.

We have that Fermi matter field equations are

$$D^2V^\alpha + m^2V^\alpha = 0 \quad (37)$$

which do not develop constraints and thus the subsidiary condition

$$D_\mu V^\mu = 0 \quad (38)$$

must be imposed to set the degrees of freedom to the 3 that define vector fields, and where the conserved quantities are given by the current

$$J_\alpha = i(V^\mu D_\alpha V_\mu^* - V_\mu^* D_\alpha V^\mu) \quad (39)$$

and by the energy

$$T_{\alpha\nu} = -g_{\alpha\nu}m^2V_\mu^*V^\mu + g_{\alpha\nu}D_\rho V_\beta^*D^\rho V^\beta - D_\alpha V_\theta^*D_\nu V^\theta - D_\alpha V^\theta D_\nu V_\theta^* \quad (40)$$

and the spin

$$S_{\rho\beta\theta} = \frac{1}{2} (V_\beta D_\rho V_\theta^* - V_\theta^* D_\rho V_\beta + V_\beta^* D_\rho V_\theta - V_\theta D_\rho V_\beta^*) \quad (41)$$

so that, whereas the condition

$$\begin{aligned} & V_\beta D_\rho V_\theta^* - V_\theta^* D_\rho V_\beta + V_\beta^* D_\rho V_\theta - V_\theta D_\rho V_\beta^* + \\ & + V_\rho D_\beta V_\theta^* - V_\theta^* D_\beta V_\rho + V_\rho^* D_\beta V_\theta - V_\theta D_\beta V_\rho^* = 0 \end{aligned} \quad (42)$$

ensures the complete antisymmetry of the spin, this form of the spin with the energy and also with the current is such that the conservation laws (11), (10) and (9) are verified.

For the counting of degrees of freedom, the subsidiary condition (38) already provides the necessary constraints that bring it to the correct amount; however, the condition of complete antisymmetry of the spin (42) has its only independent contraction given by  $V_\rho D^\rho V_\theta^* + V_\rho^* D^\rho V_\theta = 0$  after that the subsidiary condition is taken into account: this means that beyond the subsidiary condition, the conditions  $V_\rho D^\rho V_\theta^* + V_\rho^* D^\rho V_\theta = 0$ , not to mention the uncontracted condition (42), will provide further constraints that will decrease the counting of degrees of freedom to less than those that are needed to define the matter vector field.

Thus the inconsistencies of matter vector field do not regard the structure of the field equations, but the counting of the degrees of freedom, which is different but not less important a disease.

**Spinor Fields.** Finally we turn our attention to the matter spinor fields.

And as before we will begin the discussion from the vector-spinor field.

**Vector-Spinor Fields.** We consider Rarita-Schwinger spinor matter field equations as

$$i\gamma^\mu D_\mu \psi^\rho - m\sigma^{\rho\alpha} \psi_\alpha = 0 \quad (43)$$

which do not develop constraints and thus the subsidiary condition

$$\gamma_\mu \psi^\mu = 0 \quad (44)$$

must be imposed to set the degrees of freedom to the 4 that define vector-spinor fields, and where the conserved quantities are given by the current

$$J_\nu = \bar{\psi}^\alpha \gamma_\nu \psi_\alpha \quad (45)$$

and by the energy

$$T^{\sigma\rho} = \frac{i}{2} \left( \bar{\psi}^\alpha \gamma^\sigma D^\rho \psi_\alpha - D^\rho \bar{\psi}^\alpha \gamma^\sigma \psi_\alpha \right) \quad (46)$$

and the spin

$$S^{\alpha\nu\mu} = \frac{i}{2} \left[ \left( \bar{\psi}^\nu \gamma^\alpha \psi^\mu - \bar{\psi}^\mu \gamma^\alpha \psi^\nu \right) + \frac{1}{2} \bar{\psi}^\sigma \{ \gamma^\alpha, \sigma^{\nu\mu} \} \psi_\sigma \right] \quad (47)$$

so that, whereas the condition

$$\bar{\psi}^\nu \gamma^\alpha \psi^\mu - \bar{\psi}^\mu \gamma^\alpha \psi^\nu + \bar{\psi}^\nu \gamma^\mu \psi^\alpha - \bar{\psi}^\alpha \gamma^\mu \psi^\nu = 0 \quad (48)$$

ensures the complete antisymmetry of the spin, this form of the spin with the energy and also with the current is such that the conservation laws (11), (10) and (9) are verified.

Differently from the previous approach, we will follow another path to count the degrees of freedom, that is we shall not consider the subsidiary condition showing that the condition of complete antisymmetry of the spin is not entirely contained in it, but we will proceed directly to compare the constraints needed to define the vector-spinor field and those provided by the condition of complete antisymmetry of its spin: as it is known the vector-spinor field has to be defined by imposing a total number of 4 constraints upon its components, each of them being a complex field, and so a total number of 8 real constraints must be imposed, while on the other hand the condition of complete antisymmetry of the spin provides 20 real constraints, whether the subsidiary condition is considered or not, so that once again the degrees of freedom are less than those needed to define the matter vector-spinor field.

Thus inconsistencies of the Rarita-Schwinger matter vector-spinor field arise for what concerns the degrees of freedom, constituting a serious disease.

**Scalar-Spinor Fields.** Finally we have that Dirac spinor matter field equations are

$$i\gamma^\mu D_\mu \psi - m\psi = 0 \quad (49)$$

which are unconstrained, and where the conserved quantities are given by the current

$$J_\nu = \bar{\psi} \gamma_\nu \psi \quad (50)$$

and by the energy

$$T^{\sigma\rho} = \frac{i}{2} (\bar{\psi} \gamma^\sigma D^\rho \psi - D^\rho \bar{\psi} \gamma^\sigma \psi) \quad (51)$$

and the spin

$$S^{\alpha\nu\mu} = \frac{i}{4} \bar{\psi} \{\gamma^\alpha, \sigma^{\nu\mu}\} \psi \quad (52)$$

completely antisymmetric, so that this form of the spin with the energy and also with the current is such that the conservation laws (11), (10) and (9) are verified.

As it is clear, in this case we already have the complete antisymmetry of spin and no subsidiary condition is imposed, so that no constraints are provided whereas no constraints were needed anyway.

Thus the Dirac matter scalar-spinor field is unconstrained, while being perfectly defined.

## Conclusion

Throughout this paper we have outlined the discussion considering matter fields with definite matter field equations whose propagation was governed by covariant derivatives defined through connection that can be non trivial due to the presence of potential of gauge, metric and torsion.

Although it is well-known what type of issue the gauge field and the metric may create, here we have seen that it is especially the presence of torsion that create problems in the propagations of matter fields, for which the complete antisymmetry of torsion is actually able to solve part of them, but this can not be considered a general property and some of those problems remain, and even though in some cases one could try to impose fine-tuning upon the parameters or on the values of torsion, yet in some other cases a constraint was lost.

It is worth to recall that the diseases of the torsional Proca field could be cured by imposing fine-tuning upon the parameters or on the values of the torsion tensor

$$\frac{1}{6}Q^{\alpha\mu\rho}Q_{\alpha\mu\rho} \leq m^2 \quad (53)$$

and the diseases of the torsional Rarita-Schwinger field for the gravitino lead to the loss of constraints; curiously the diseases of the torsionless Rarita-Schwinger field for the gravitino can be cured by assuming some fine-tuning upon parameters as explained in [6] or on the value of fields

$$\left| \frac{2e\vec{B}}{3m} \right|^2 \leq m^2 \quad (54)$$

as shown in [5] whereas the disease encountered for the torsionless spin-2 field is the loss of constraints as shown in [4]: roughly speaking, the problems are quite in general the same for the torsional spin- $s$  field as for the torsionless spin- $(s + \frac{1}{2})$  field. It is as tough torsion would account for an additional spin- $\frac{1}{2}$  component in exacerbating the problems discussed above.

On the other hand, the complete antisymmetry of the spin has also been considered as a constraint that could alter the balance of degrees of freedom needed to define a given matter field, resulting in the fact that such a condition provides constraints that are always more than needed, or else that the given matter field is always more restricted than necessary.

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